Essential Mathematics for Neuroscience
Lecture 3 - Derivatives

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Definition of Derivative

The derivative $f'(x_0) = \frac{df}{dx}(x_0)$ of a function has two intuitive meanings:
1. It measures the rate of change of a function at a certain location.
2. It is the slope of the line touching the function at a the point. This line is called tangent line or simply tangent.
Derivative in terms of limits

\[ f'(x) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = x_1 - x_0 \]
Derivative in terms of limits

$$f'(x) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Differential quotient
A function $f$ is said to be continuous at a point $x_0$ if
\[ \lim_{h \to 0} f(x_0 + h) = \lim_{h \to 0} f(x_0 - h) = f(x_0), \]
i.e. if getting infinitesimal close to $x_0$ (no matter from which side) implies getting infinitesimal close to $f(x_0)$.
A function is said to be continuous, if it is continuous in every point of its domain.
Differentiable function

A function $f$ is said to be differentiable at a point $x_0$ if there exists a linear function $L(x)$ such that
$$L(x_0) - \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = 0,$$
no matter from which side $h$ approaches zero.

A function $f$ is said to be differentiable, if it is differentiable in every point of its domain.
Differentiation rules

Basic rules

- Derivatives of constant functions: The derivative of any constant function $f(x) = a$ is $f'(x) = 0$.

- Summation Rule: Let $f(x) = \sum_{k=1}^{n} g_k(x)$ a sum of arbitrary differentiable functions. Then $f'(x)$ is given by:
  \[ f'(x) = \sum_{k=1}^{n} g'_k(x). \]

- Power Rule: Let $f(x) = ax^b$ be, then is give by:
  \[ f'(x) = abx^{b-1}. \]
Differentiation rules

Combination rules

- **Chain Rule:** Let \( f(x) = g_2(g_1(x)) \) be a composition of arbitrary differentiable functions. Then \( f'(x) \) is given by:
  \[
  f'(x) = g_2'(g_1(x)) \cdot g_1'(x).
  \]

- **Product Rule:** Let \( f(x) = g_1(x)g_2(x) \) be a product of arbitrary differentiable functions. Then \( f'(x) \) is given by:
  \[
  f'(x) = g_1'(x) \cdot g_2(x) + g_1(x) \cdot g_2'(x).
  \]

- **Quotient Rule:** Let \( f(x) = \frac{g_1(x)}{g_2(x)} \) be a quotient of two arbitrary differentiable functions. Then \( f'(x) \) is given by:
  \[
  f'(x) = \frac{g_1'(x)g_2(x) - g_1(x)g_2'(x)}{g_2(x)^2}.
  \]
Derivatives of important functions

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<thead>
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<th>Derivatives</th>
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<tbody>
<tr>
<td>• $f(x) = \sin(x) \Rightarrow f'(x) = \cos(x)$</td>
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<tr>
<td>• $f(x) = \cos(x) \Rightarrow f'(x) = -\sin(x)$</td>
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<tr>
<td>• $f(x) = e^x \Rightarrow f'(x) = e^x$</td>
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<tr>
<td>• $f(x) = \ln(x) = \log_e(x) \Rightarrow f'(x) = \frac{1}{x}$</td>
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